Stability Results on Coupled Fixed Point Iterative Procedures in Complete Metric Spaces

S.A. Aniki and K. Rauf
Department of Mathematics, University of Ilorin, Nigeria
smlaniki@yahoo.com

Abstract
The concept of coupled fixed point $T: X^2 \to X$ has attracted several researches in recent time, both in partially ordered metric spaces and in cone metric spaces.In this work we introduce the notion of stability definition of coupled fixed point iteration procedures$s_{n+1} = T(s_n, t_n)$, $t_{n+1} = T(t_n, s_n), n \geq 0$, with $\{(s_n, t_n)\} \subset X^2$, for $s_0, t_0 \in X^2$ in partially ordered set $(X, \leq)$ and establish stability results for mixed monotone mappings satisfying various contractive conditions. Our results extend and complete some existing results in the literature.

Keywords: Coupled fixed point; stability; partially ordered; mixed monotone; mapping.

Introduction

It is well known that the Banach contraction principle plays an important role in nonlinear analysis. It is often used to solve integral equation, differential equation and periodic boundary value problems. Over the past decades, the famous principle has been generalized in several directions. Guo and Lakshmikantham (2009) introduced the concept of the coupled fixed point. Consequently, Bhaskar and Lakchmikantham (2006) required the coupled fixed-point theorem in partially ordered metric spaces. Lakchmikantham and Ćirić (2009) presented the mixed g −monotone mapping in the partially ordered metric spaces for the first place. Later, Borcut and Berinde (2012) generalized their result and got the tripled coincidence theorem. In order to compensate the restriction in common metric spaces, some experts have popularized the Banach contraction principle on more generalized metric spaces. The existence of a fixed point for contraction type mappings in partially ordered metric spaces has been considered (Agarwal *et al.*, 2008) and (Nieto and López, 2005 & 2007). In Van Loung & Xuan Thuan (2011) and Bhaskar and Lakchmikantham (2006), the authors proved some coupled fixed-point theorems and noted their results can be used to investigate a large class of problems and have discussed the existence and uniqueness of solution for a periodic boundary value problem and a nonlinear integral equation. Samet and Vetro (2010) introduced the notion of fixed point of N-order as a natural extension of coupled fixed point and established some new coupled fixed-point theorems in complete metric spaces. Very recently, Berinde and Pacurar (2015) use a constructive approach to coupled fixed-point theorems in metric spaces. Some work on fixed points, and the stability of a fixed point iterative procedures was first studied by Ostrowski (1967) in the case of Banach contraction mappings and this subject was later developed for certain contractive definitions by several authors, Harder and Hicks (1988), Rhoades (1990 & 1993), Osilike (1996-1995), Osilike and Udomene (1995), Olatinwo (2010), Imoru and Olatinwo (2006), Timis (2014), Imoru, Olatinwo and Owojori (2006). Our aim in this work is to show the concept of stability for coupled fixed point iteration procedures and to establish stability results for mixed monotone mappings satisfying various contractive conditions by extension from constructive approach to coupled fixed point theorems in metric spaces by Berinde and Pacurar (2015).

Preliminaries

Definition 1: Let X be a non-empty set. A mapping $d: X \times X \to \mathbb{R}$ (the set of reals) is said to be a metric (or distance function) if and only if d satisfies the following axioms:

i. $d(x, y) \geq 0$ for all $x, y \in X$,
ii. $d(x, y) = 0$ if and only $x = y$,
iii. $d(x, y) = d(y, x)$ for all $x, y \in X$,
iv. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d is metric for X, then the ordered pair (X, d) is called a metric space and $d(x, y)$ is called the distance between x and y.

Definition 2: (Agarwal *et al.*, 2008)A self-mapping T of a metric space (X, d) is said to be Lipschitzian if for all $x, y \in X$ and $\alpha \geq 0$

$$d\big(T(x), T(y)\big) \leq \alpha d(x, y)$$

T is said to be contraction on α if $\alpha \in [0,1)$ and non-expansive if $\alpha = 1$.

Definition 3: Let X be a nonempty setand d a metric on X so that the pair (X, d) is a Cauchy space and let $\{x_n\}$ be a sequence of points in X, then it is said to be a Cauchy sequence in X if and only if for every $\epsilon > 0$ there exists a positive integer N such that, $m, n \geq N \Rightarrow d(x_m, x_n) < \epsilon$.

Definition 4 (Matthews, 1994) A partial metric on a nonempty set $X$is a function $d: X^2 \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

i. $x = y \Leftrightarrow d(x, x) = d(x, y) = d(y, y)$,
ii. $d(x, x) \leq d(x, y)$,
iii. $d(x, y) = d(y, x)$,
iv. $d(x, y) \leq d(x, z) + d(z, y) - d(z, z)$

A partial metric space is a pair (X, d) such that X is nonempty set and d is a partial metric on X.

Definition 5: (Agarwal et al., 2008)Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Then, the product space X^2 has the following partial order

$$(q, r) \leq (s, t) \Leftrightarrow s \geq q, t \leq r; \quad (q, r), (s, t) \in X^2$$

Definition 6: (Berinde and Parcurar, 2015) Let $(X, \leq)$ be a partially ordered set and $T: X^2 \to X$ be a mapping. We say that T has a mixed monotone property if $T(s, t)$ is monotone nondecreasing in s, and monotone nonincreasing in t, that is for any $s, t \in X$,

$$s_1 \leq s_2 \Rightarrow T(s_1, t) \leq T(s_2, t), \quad s_1, s_2 \in X,$$
$$t_1 \leq t_2 \Rightarrow T(s, t_1) \geq T(s, t_2), \quad t_1, t_2 \in X,$$

Definition 7: (Berinde and Parcurar, 2015) An element $(s, t) \in X^2$ is called coupled fixed point of $T: X^2 \to X$, if

$$T(s, t) = s, \qquad T(t, s) = t.$$

Definition 8: (Berinde and Parcurar, 2015) A mapping $T: X^2 \to X$ is said to be (κ, μ)-contraction if and only if there exist two constants $\kappa \geq 0, \mu \geq 0, \kappa + \mu < 1$, such that $\forall\ s, t, q, r \in X$,

$$d\big(T(s, t), T(q, r)\big) \leq \kappa d(s, q) + \mu d(t, r)$$

In order to prove our main stability result in this work, we give the followings:

Definition 9: A mapping $T: X^2 \to X$ is said to be (κ, μ)-contraction if and only if there exist two constants $\kappa \geq 0, \mu \geq 0, \kappa + \mu < 1$, such that $\forall\ s, t, q, r \in X$,

$$d\big(T(s, t), T(q, r)\big) \leq \kappa d(s, q) + \mu d(t, r) \tag{1}$$

From (1) above, we introduce some new contractive conditions.

Let (X, d) be a metric space. For a map $T: X^2 \to X$ there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, with $\alpha_1 + \alpha_2 < 1, \beta_1 + \beta_2 < 1$, such that $\forall s, t, q, r \in X$ we introduce the following definitions of contractive conditions:

i.
$$d\big(T(s,t), T(q,r)\big) \leq \alpha_1 d(T(s,t), s) + \beta_1 d(T(q,r), q); \quad (2)$$
$$d\big(T(t,s), T(r,q)\big) \leq \alpha_2 d(T(t,s), t) + \beta_2 d(T(r,q), r); \quad (3)$$

ii.
$$d\big(T(s,t), T(q,r)\big) \leq \alpha_1 d(T(s,t), q) + \beta_1 d(T(q,r), s); \quad (4)$$
$$d\big(T(t,s), T(r,q)\big) \leq \alpha_2 d(T(t,s), r) + \beta_2 d(T(r,q), t); \quad (5)$$

Let $A, B \in M_{(m,n)}(\mathbb{R})$ be two matrices. We write $A \leq B$, if $a_{ij} \leq b_{ij}$ for all $i = \overline{1,m},\ j = \overline{1,n}$.

Lemma 1: Let $\{a_n\}, \{b_n\}$ be sequences of non-negative numbers and h be a constant, such that $0 \leq h < 1$ and $a_{n+1} \leq h a_n + b_n, n \geq 0$. If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

We also give the following result which extends Lemma 1 to vector sequences, where inequalities between vectors mean inequality on its elements.

Lemma 2: Let $\{q_n\}, \{r_n\}$ be sequences of nonnegative real numbers, consider a matrix $A \in M_{(2,2)}(\mathbb{R})$ with nonnegative elements, such that

$$\begin{pmatrix} q_{n+1} \\ r_{n+1} \end{pmatrix} \leq A. \begin{pmatrix} q_n \\ r_n \end{pmatrix} + \begin{pmatrix} \delta_n \\ \gamma_n \end{pmatrix}, \ \ n \geq 0, \quad (6)$$

with

i. $\lim_{n\to\infty} A^n = 0_2$,
ii. $\sum_{k=0}^{\infty} \delta_n < \infty$ and $\sum_{k=0}^{\infty} \gamma_n < \infty$.

If $\lim_{n\to\infty} \begin{pmatrix} \delta_n \\ \gamma_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $\lim_{n\to\infty} \begin{pmatrix} q_n \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Proof:
For $A = 0 \in M_{(2,2)}$, we shall rewrite (6) with $n = k$ to obtain the following inequalities for $k = 0,1,2, \dots, n$.

$$At\ k = 0: \quad \begin{pmatrix} q_1 \\ r_1 \end{pmatrix} \leq A \cdot \begin{pmatrix} q_0 \\ r_0 \end{pmatrix} + \begin{pmatrix} \delta_0 \\ \gamma_0 \end{pmatrix} \quad (7)$$

$$At\ k = 1: \quad \begin{pmatrix} q_2 \\ r_2 \end{pmatrix} \leq A \cdot \begin{pmatrix} q_1 \\ r_1 \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \gamma_1 \end{pmatrix} \quad (8)$$

$$At\ k = 2: \quad \begin{pmatrix} q_3 \\ r_3 \end{pmatrix} \leq A \cdot \begin{pmatrix} q_2 \\ r_2 \end{pmatrix} + \begin{pmatrix} \delta_2 \\ \gamma_2 \end{pmatrix} \quad (9)$$

⋮

$$At\ k = n-1: \quad \begin{pmatrix} q_n \\ r_n \end{pmatrix} \leq A \cdot \begin{pmatrix} q_{n-1} \\ r_{n-1} \end{pmatrix} + \begin{pmatrix} \delta_{n-1} \\ \gamma_{n-1} \end{pmatrix} \quad (10)$$

$$At\ k = n: \quad \begin{pmatrix} q_{n+1} \\ r_{n+1} \end{pmatrix} \leq A \cdot \begin{pmatrix} q_n \\ r_n \end{pmatrix} + \begin{pmatrix} \delta_n \\ \gamma_n \end{pmatrix} \quad (11)$$

Now, the sum of the inequalities is as follows:

Since,
If $\lim_{n\to\infty} \begin{pmatrix} \delta_n \\ \gamma_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $\lim_{n\to\infty} \begin{pmatrix} q_n \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

we obtain

$$\binom{q_{n+1}}{r_{n+1}} \le A^{n+1} \cdot \binom{q_0}{r_0} + \sum_{k=0}^{n} A^k \cdot \binom{\delta_{n-k}}{\gamma_{n-k}}. \tag{12}$$

From eqn. 6(ii), it follows that the sequence of partial sums $\{\Delta_n\}$ and $\{\Gamma_n\}$, given by $\Delta_n = \delta_0 + \delta_1 + \cdots + \delta_n$ and $\Gamma_n = \gamma_0 + \gamma_1 + \cdots + \gamma_{n,}$ for $n \ge 0$, converges respectively to some $\Delta_n \ge 0$ and $\Gamma_n \ge 0$ and hence they are bounded.

Let $M > 0$ be such that $\binom{\Delta_n}{\Gamma_n} \le M.\binom{1}{1}, \quad \forall \ \ n \ge 0$.By eqn. 6(i) we have that $\forall \ \ e > 0$, there exists $N = N(e)$ such that $A^n \le \frac{e}{2M} \cdot I_2, \ \ \forall \ \ n \ge N, \ \ M > 0$.

We can write

$$\sum_{k=0}^{n} A^k \binom{\delta_{n-k}}{\gamma_{n-k}} = A^n \binom{\delta_0}{\gamma_0} + \cdots + A^N \binom{\delta_{n-N}}{\gamma_{n-N}} + A^{N-1} \binom{\delta_{n-N+1}}{\gamma_{n-N+1}} + \cdots + I_2 \binom{\delta_n}{\gamma_n}$$

but

$$A^n \binom{\delta_0}{\gamma_0} + \cdots + A^N \binom{\delta_{n-N}}{\gamma_{n-N}} \le \frac{e}{2M} \cdot I_2 \left[\binom{\delta_0}{\gamma_0} + \cdots + \binom{\delta_{n-N}}{\gamma_{n-N}}\right]$$

$$= \frac{e}{2M} \cdot I_2 \binom{\Delta_{n-N}}{\Gamma_{n-N}} \le \frac{e}{2M} \cdot I_2 \cdot M \binom{1}{1} = \frac{e}{2}\binom{1}{1}$$

for all $n \ge N$. Similarly, if we denote $A' = max\{I_2, A, \cdots, A^{N-1}\}$, we obtain

$$A^{N-1} \binom{\delta_{n-N+1}}{\gamma_{n-N+1}} + \cdots + I_2 \binom{\delta_n}{\gamma_n} \le A' \left[\binom{\delta_{n-N+1}}{\gamma_{n-N+1}} + \cdots + \binom{\delta_n}{\gamma_n}\right] = A' \binom{\Delta_n - \Delta_{n-N}}{\Gamma_n - \Gamma_{n-N}}$$

As N is fixed, then $\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} \Delta_{n-N} = \Delta$, and $\lim_{n\to\infty} \Gamma_n = \lim_{n\to\infty} \Gamma_{n-N} = \Gamma$,

which shows that there exists a positive integer K such that

$$A' \binom{\Delta_n - \Delta_{n-N}}{\Gamma_n - \Gamma_{n-N}} < \frac{e}{2}\binom{1}{1}, \ \ \forall \ \ n \ge K.$$

Now, for $m = max\{K, N\}$, we get

$$A^n \binom{\delta_0}{\gamma_0} + \cdots + I_2 \binom{\delta_n}{\gamma_n} < e\binom{1}{1}, \ \ \forall \ \ n \ge n,$$

and therefore,

$$\lim_{n\to\infty} \sum_{k=0}^{n} A^k \binom{\delta_{n-k}}{\gamma_{n-k}} = 0.$$

Now, by letting limit in (12), and $\lim_{n\to\infty} A^n = 0$, we obtain

$$\lim_{n\to\infty} \binom{q_n}{r_n} = \binom{0}{0},$$

as required.

Stability Results

Let (X,d) be a metric space and $T: X^2 \to X$ a mapping. For $(s_0,t_0) \in X^2$ the sequence $\{(s_n,t_n)\} \subset X^2$ defined by

$$s_{n+1} = T(s_n,t_n), \;\; t_{n+1} = T(t_n,s_n) \tag{13}$$

for $n = 0,1,2,...$, is said to be coupled fixed point iterative procedures.

We give the following stability definition with respect to T, in metric spaces, relative to tripled fixed points iterative procedures.

Definition 10: Let (X,d) be a complete metric space and $Fix_c(T) = \{(s^*,t^*) \in X^2 | T(s^*,t^*) = s^*, T(t^*,s^*) = t^*\}$ is the set of coupled fixed points of T.

Let $\{(s_n,t_n)\} \subset X^2$ be the sequence generated by the iterative procedure defined by (13), where $(s_0,t_0) \in X^2$ is the initial value, which converges to a coupled fixed point (s^*,t^*) of T.

Let $(q_n,r_n) \subset X^2$ be an arbitrary sequence. For all $n = 0,1,2,...$, we set

$$\delta_n = d\big(q_{n+1}, T(q_n,r_n)\big), \qquad \gamma_n = d\big(r_{n+1}, T(r_n,q_n)\big).$$

Then, the coupled fixed-point iterative procedure defined by (13) is T −stable or stable with respect to T, if and only if

$$\lim_{n\to\infty}(\delta_n,\gamma_n) = 0_{\mathbb{R}^2}$$
$$\Longrightarrow \lim_{n\to\infty}(q_n,r_n) = (s^*,t^*).$$

Theorem 1: Let $(X,\leq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T: X^2 \to X$ be a continuous mapping having a mixed monotone property on X and satisfying the contraction (1).

If there exists $s_0, t_0 \in X$ such that

$$s_0 \leq T\big(s_{0,}t_0\big) \; and \; t_0 \geq T(t_0,s_0)$$

then, there exist $s^*, t^* \in X$ such that

$$s^* = T(s^*,t^*) \text{ and } t^* = T(t^*,s^*).$$

Assume that for every $(s,t), (s_1,t_1) \in X^2$, then there exists $(q,r), (q_1,r_1) \in X^2$ that is comparable to (s,t) and (s_1,t_1). For $(s_0,t_0) \in X^2$, let $\{(s_n,t_n)\} \subset X^2$ be the fixed point iterative procedure defined by (13). Then, the coupled fixed-point iterative procedure is stable with respect to T.

Proof:

From the suppositions of the hypothesis, Berinde and Borcut (2012) proved the existence and the uniqueness of fixed point, now we will study the stability of coupled fixed point iterative procedures.

Let $\{(s_n,t_n)\} \subset X^2$, $\;\delta_n = d\big(q_{n+1}, T(q_n,r_n)\big)$, $\;\gamma_n = d\big(r_{n+1}, T(r_n,q_n)\big)$. Assuming also that

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\gamma_n = 0,$$

in order to establish that

$$\lim_{n\to\infty} q_n = s^* \; and \; \lim_{n\to\infty} r_n = t^*.$$

Therefore, using (κ,μ) −contraction in condition (1), we have

$$\begin{aligned} d(q_{n+1},s^*) &\le d\big(q_{n+1},T(q_n,r_n)\big) + d(T(q_n,r_n),s^*) \\ &= d(T(q_n,r_n),T(s^*,t^*)) + \delta_n \\ &\le \kappa d(q_n,s^*) + \mu d(r_n,t^*) + \delta_n \end{aligned} \tag{14}$$

$$\begin{aligned} d(r_{n+1},t^*) &\le d\big(r_{n+1},T(r_n,q_n)\big) + d(T(r_n,q_n),t^*) \\ &= d(T(r_n,q_n),T(t^*,s^*)) + \gamma_n \\ &\le \kappa d(r_n,t^*) + \mu d(q_n,s^*) + \gamma_n, \end{aligned} \tag{15}$$

From (14) and (15), we have

$$\begin{pmatrix} d(q_{n+1},s^*) \\ d(r_{n+1},t^*) \end{pmatrix} \le \begin{pmatrix} \kappa & \mu \\ \mu & \kappa \end{pmatrix} \cdot \begin{pmatrix} d(q_n,s^*) \\ d(r_n,t^*) \end{pmatrix} + \begin{pmatrix} \delta_n \\ \gamma_n \end{pmatrix}$$

We denote $A \coloneqq \begin{pmatrix} \kappa & \mu \\ \mu & \kappa \end{pmatrix}$, where $0 \le \kappa + \mu < 1$, from (1).

On applying Lemma 2, we need that $A^n \to 0$ as $n \to \infty$.By a way of simplification, we write

$$A \coloneqq \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

where

$$a_1 + b_1 = c_1 + d_1 = \kappa + \mu < 1.$$

Then,

$$\begin{aligned} A^2 &= \begin{pmatrix} \kappa & \mu \\ \mu & \kappa \end{pmatrix} \cdot \begin{pmatrix} \kappa & \mu \\ \mu & \kappa \end{pmatrix} \\ &= \begin{pmatrix} \kappa^2 + \mu^2 & 2\kappa\mu \\ 2\kappa\mu & \kappa^2 + \mu^2 \end{pmatrix} \coloneqq \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \end{aligned}$$

where

$$a_2 + b_2 = c_2 + d_2 = \kappa^2 + \mu^2 + 2\kappa\mu = (\kappa + \mu)^2 < \kappa + \mu < 1$$

Now, on proving by induction that

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

where

$$a_n + b_n = c_n + d_n = (\kappa + \mu)^n < \kappa + \mu < 1. \tag{16}$$

If we assume that (16) is true for n, then

$$\begin{aligned} A^{n+1} &= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \cdot \begin{pmatrix} \kappa & \mu \\ \mu & \kappa \end{pmatrix} \\ &= \begin{pmatrix} \kappa a_n + \mu b_n & \mu a_n + \kappa b_n \\ \kappa c_n + \mu d_n & \mu c_n + \kappa d_n \end{pmatrix}, \end{aligned}$$

we have

$$\begin{aligned} a_{n+1} + b_{n+1} &= \kappa a_n + \mu b_n + \mu a_n + \kappa b_n \\ &= (\kappa + \mu) a_n + (\kappa + \mu) b_n \\ &= (\kappa + \mu)(a_n + b_n) \end{aligned}$$

From (16), we have

$$\begin{aligned} &= (\kappa + \mu)(\kappa + \mu)^n \\ &= (\kappa + \mu)^{n+1} < \kappa + \mu < 1 \end{aligned}$$

Similarly,

$$C_{n+1} + d_{n+1} = (\kappa + \mu)^{n+1} < \kappa + \mu < 1.$$

Therefore,

$$\lim_{n\to\infty} A^n = 0_2$$

Now, having satisfied the conditions of the hypothesis of Lemma 2, on applying we get

$$\lim_{n\to\infty} \binom{q_n}{r_n} = \binom{s^*}{t^*},$$

So, the coupled fixed-point iteration procedure defined by (13) is T −stable.

Corollary 1:
Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X^2 \to X$ be a continuous mapping having a mixed monotone property on X.

There exists $h \in [0,1)$, such that T satisfies the following contraction condition.

$$d\big(T(s,t), T(q,r)\big) \leq \frac{h}{2}[d(s,q) + d(t,r)], \tag{17}$$

for each $s, t, q, r \in X$, with $s \geq q$ and $t \leq r$.

If there exists $s_0, t_0 \in X$ such that

$$s_0 \leq T\big(s_{0,} t_0\big) \; and \; t_0 \geq T(t_0, s_0)$$

Then, there exist $s^*, t^* \in X$ such that

$$s\hat{} *= T(s\hat{} *, t\hat{} *) \; and \; t\hat{} *= T(t\hat{} *, s\hat{} *).$$

Assume that for every $(s,t), (s_1, t_1) \in X^2$, then there exists $(q,r), \in X^2$ that can be compared to (s,t) and (s_1, t_1). For $(s_0, t_0) \in X^2$, let $\{(s_n, t_n)\} \subset X^2$ be a coupled fixed point iterative procedure defined by (13). Then, the coupled fixed-point iterative procedure is stable with respect to T.

Proof:

On applying theorem 1, for $\kappa = \mu := \frac{h}{2}$.Let $\{(s_n, t_n)\} \subset X^2$, $\delta_n = d\big(q_{n+1}, T(q_n, r_n)\big)$, $\gamma_n = d\big(r_{n+1}, T(r_n, q_n)\big)$. Assuming also that $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$, to be able to establish that $\lim_{n\to\infty} q_n = s^* \; and \; \lim_{n\to\infty} r_n = t^*$.

Therefore, using contraction in condition (17), we have

$$\begin{aligned} d(q_{n+1}, s^*) &\le d\big(q_{n+1}, T(q_n, r_n)\big) + d(T(q_n, r_n), s^*) \\ &= d(T(q_n, r_n), T(s^*, t^*)) + \delta_n \\ &\le \frac{h}{2} d(q_n, s^*) + \frac{h}{2} d(r_n, t^*) + \delta_n \end{aligned} \tag{18}$$

$$\begin{aligned} d(r_{n+1}, t^*) &\le d\big(r_{n+1}, T(r_n, q_n)\big) + d(T(r_n, q_n), t^*) \\ &= d(T(r_n, q_n), T(t^*, s^*)) + \gamma_n \\ &\le \frac{h}{2} d(r_n, t^*) + \frac{h}{2} d(q_n, s^*) + \gamma_n, \end{aligned} \tag{19}$$

From (18) and (19), we have

$$\begin{pmatrix} d(q_{n+1}, s^*) \\ d(r_{n+1}, t^*) \end{pmatrix} \le \begin{pmatrix} \frac{h}{2} & \frac{h}{2} \\ \frac{h}{2} & \frac{h}{2} \end{pmatrix} \cdot \begin{pmatrix} d(q_n, s^*) \\ d(r_n, t^*) \end{pmatrix} + \begin{pmatrix} \delta_n \\ \gamma_n \end{pmatrix}$$

We denote $A := \begin{pmatrix} \frac{h}{2} & \frac{h}{2} \\ \frac{h}{2} & \frac{h}{2} \end{pmatrix}$, where $0 \le \frac{h}{2} + \frac{h}{2} = h < 1$,

On applying Lemma 2, we need that $A^n \to 0$, as $n \to \infty$.By simplification, we write

$$A := \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

where

$$a_1 + b_1 = c_1 + d_1 = \frac{h}{2} + \frac{h}{2} = h < 1.$$

Then,

$$\begin{aligned} A^2 &= \begin{pmatrix} \frac{h}{2} & \frac{h}{2} \\ \frac{h}{2} & \frac{h}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{h}{2} & \frac{h}{2} \\ \frac{h}{2} & \frac{h}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{h^2}{2} & \frac{h^2}{2} \\ \frac{h^2}{2} & \frac{h^2}{2} \end{pmatrix} := \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \end{aligned}$$

$$a_2 + b_2 = c_2 + d_2 = \frac{h^2}{2} + \frac{h^2}{2} = h^2 < h < 1.$$

Then,

$$\begin{aligned} A^3 &= A^2 \cdot A \\ &= \begin{pmatrix} \frac{h^2}{2} & \frac{h^2}{2} \\ \frac{h^2}{2} & \frac{h^2}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{h}{2} & \frac{h}{2} \\ \frac{h}{2} & \frac{h}{2} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{h^3}{2} & \frac{h^3}{2} \\ \frac{h^3}{2} & \frac{h^3}{2} \end{pmatrix} \coloneqq \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$$

where,

$$a_3 + b_3 = c_3 + d_3 = \frac{h^3}{2} + \frac{h^3}{2} = h^3 < h^2 < h < 1$$

Now, on proving by induction that

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

where,

$$a_n + b_n = c_n + d_n = \frac{h^n}{2} + \frac{h^n}{2} = h^n < h^{n-1} < \cdots < h^2 < h < 1 \tag{20}$$

If we assume that (20) is true for n, then

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \cdot \begin{pmatrix} \frac{h}{2} & \frac{h}{2} \\ \frac{h}{2} & \frac{h}{2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{ha_n}{2} + \frac{hb_n}{2} & \frac{ha_n}{2} + \frac{hb_n}{2} \\ \frac{hc_n}{2} + \frac{hd_n}{2} & \frac{hc_n}{2} + \frac{hd_n}{2} \end{pmatrix} \coloneqq \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix}$$

we have

$$\begin{aligned} a_{n+1} + b_{n+1} &= \frac{ha_n}{2} + \frac{hb_n}{2} + \frac{ha_n}{2} + \frac{hb_n}{2} \\ &= ha_n + hb_n \\ &= h(a_n + b_n) \end{aligned}$$

From (20), we have

$$= h(h^n) = h^{n+1} < \cdots < h < 1$$

Similarly,

$$c_{n+1} + d_{n+1} = h(h^n) = h^{n+1} < \cdots < h < 1$$

Therefore,

$$\lim_{n \to \infty} A^n = 0_2$$

Now, having satisfied the conditions of the hypothesis of Lemma 2, we apply to get

$$\lim_{n \to \infty} \begin{pmatrix} q_n \\ r_n \end{pmatrix} = \begin{pmatrix} s^* \\ t^* \end{pmatrix},$$

Then, the coupled fixed-point iteration procedure is stable with respect to T.

Theorem 2: Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X^2 \to X$ be a continuous mapping having a mixed monotone property on X satisfying (2) and (3).

If there exists $s_0, t_0 \in X$ such that

$$s_0 \leq T(s_{0,} t_0) \text{ } and \text{ } t_0 \geq T(t_0, s_0),$$

then, there exist $s^*, t^* \in X$ such that

$$s\hat{} *= T(s\hat{} *, t\hat{} *) \text{ } and \text{ } t\hat{} *= T(t\hat{} *, s\hat{} *).$$

Assume that for every $(s, t), (s_1, t_1) \in X^2$, then there exists $(q, r), \in X^2$ that can be compared to (s, t) and (s_1, t_1). For $(s_0, t_0) \in X^2$, let $\{(s_n, t_n)\} \subset X^2$ be a coupled fixed point iterative procedure defined by (13). Then, the coupled fixed-point iterative procedure is stable with respect to T.

Proof:

Let $\{(s_n, t_n)\} \subset X^2$, $\delta_n = d\big(q_{n+1}, T(q_n, r_n)\big)$, $\gamma_n = d\big(r_{n+1}, T(r_n, q_n)\big)$. Assuming that $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$, in order to establish that $\lim_{n\to\infty} q_n = s^* \text{ } and \text{ } \lim_{n\to\infty} r_n = t^*$.

Therefore, using contraction condition (2), we have

$$\begin{aligned}
d(q_{n+1}, s^*) &\leq d\big(q_{n+1}, T(q_n, r_n)\big) + d(T(q_n, r_n), s^*) \\
&= d\big(T(q_n, r_n), T(s^{\cdot}, t^{\cdot})\big) + \delta_n \\
&\leq \alpha_1 d(T(s^*, t^*), s^*) + \beta_1 d(T(q_n, r_n), q_n) + \delta_n \\
&\leq \alpha_1 d(s^*, s^*) + \beta_1 d(T(q_n, r_n), q_{n+1}) + \beta_1 d(q_{n+1}, s^*) + \beta_1 d(s^*, q_n) + \delta_n \\
&= \alpha_1 d(s^*, s^*) + \beta_1 d(q_{n+1}, s^*) + \beta_1 d(s^*, q_n) + \beta_1 \delta_n + \delta_n \\
&= \alpha_1 d(s^*, s^*) + \beta_1 d(q_{n+1}, s^*) + \beta_1 d(s^*, q_n) + (\beta_1 + 1)\delta_n \\
d(q_{n+1}, s^*) - \beta_1 d(q_{n+1}, s^*) &= \alpha_1 d(s^*, s^*) + \beta_1 d(s^*, q_n) + (\beta_1 + 1)\delta_n \\
(1 - \beta_1) d(q_{n+1}, s^*) &= \beta_1 d(s^*, q_n) + (\beta_1 + 1)\delta_n + \alpha_1 d(s^*, s^*)
\end{aligned}$$

Hence,

$$(1 - \beta_1) d(q_{n+1}, s^*) \leq \beta_1 d(s^*, q_n) + \delta_n' \qquad (21)$$

where

$$\delta_n' \coloneqq (\beta_1 + 1)\delta_n + \alpha_1 d(s^*, s^*).$$

On applying Lemma 1 on (21), we have

$$d(q_{n+1}, s^*) \leq \frac{\beta_1}{(1 - \beta_1)} d(s^{\cdot}, q_n) + \frac{\delta_n'}{(1 - \beta_1)},$$

for $\frac{\beta_1}{(1-\beta_1)} \in [0,1)$, we obtain that $\lim_{n\to\infty} q_n = s^*$.

Similarly, on using the contraction condition (3), we obtain

$$\begin{aligned}
d(r_{n+1}, t^*) &\leq d\big(r_{n+1}, T(r_n, q_n)\big) + d(T(r_n, q_n), t^*) \\
&= d\big(T(r_n, q_n), T(t^*, s^*)\big) + \gamma_n \\
&\leq \alpha_2 d(T(t^*, s^*), t^*) + \beta_2 d(T(r_n, q_n), r_n) + \gamma_n
\end{aligned}$$

$$\leq \alpha_2 d(t^*,t^*) + \beta_2 d(T(r_n,q_n),r_{n+1}) + \beta_2 d(r_{n+1},t^*) + \beta_2 d(t^*,r_n) + \gamma_n$$
$$= \alpha_2 d(t^*,t^*) + \beta_2 d(r_{n+1},t^*) + \beta_2 d(t^*,r_n) + \beta_2\gamma_n + \gamma_n$$
$$= \alpha_2 d(t^*,t^*) + \beta_2 d(r_{n+1},t^*) + \beta_2 d(t^*,r_n) + (\beta_2+1)\gamma_n$$
$$d(r_{n+1},t^*) - \beta_2 d(r_{n+1},t^*) = \alpha_2 d(t^*,t^*) + \beta_2 d(t^*,r_n) + (\beta_2+1)\gamma_n$$
$$(1-\beta_2)d(r_{n+1},t^*) = \beta_2 d(t^*,r_n) + (\beta_2+1)\gamma_n + \alpha_2 d(t^*,t^*)$$

Hence,

$$(1-\beta_2)d(r_{n+1},t^*) \leq \beta_2 d(t^*,r_n) + \gamma_n' \quad (22)$$

where

$$\gamma_n' \coloneqq (\beta_2+1)\gamma_n + \alpha_2 d(t^*,t^*).$$

On applying Lemma 1 on (22), we have

$$d(r_{n+1},t^*) \leq \frac{\beta_2}{(1-\beta_2)} d(t^*,r_n) + \frac{\gamma_n'}{(1-\beta_2)},$$

for $\frac{\beta_2}{(1-\beta_2)} \in [0,1)$, we obtain that $\lim_{n\to\infty} r_n = t^*$,

which is the conclusion.

Theorem 3: Let $(X,\leq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T: X^2 \to X$ be a continuous mapping having a mixed monotone property on X satisfying (4) and (5).

If there exists $s_0, t_0 \in X$ such that

$$s_0 \leq T(s_{0,}t_0) \; and \; t_0 \geq T(t_0,s_0),$$

then, there exists $s^*, t^* \in X$ such that

$$s^\wedge *= T(s^\wedge *, t^\wedge *) \; and \; t^\wedge *= T(t^\wedge *, s^\wedge *).$$

Assume that for every $(s,t),(s_1,t_1) \in X^2$, then there exists $(q,r), \in X^2$ that can be compared to (s,t) and (s_1,t_1). For $(s_0,t_0) \in X^2$, let $\{(s_n,t_n)\} \subset X^2$ be a coupled fixed point iterative procedure defined by (13). Then, the coupled fixed-point iterative procedure is stable with respect to T.

Proof

Let $\{(s_n,t_n)\}_{n=0}^{\infty} \subset X^2$, $\delta_n = d(q_{n+1}, T(q_n,r_n))$, $\gamma_n = d(r_{n+1}, T(r_n,q_n))$. Assume also that $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$, to be able to establish that $\lim_{n\to\infty} q_n = s^*$and $\lim_{n\to\infty} r_n = t^*$.

Therefore, using contraction condition (4), we obtain

$$d(q_{n+1},s^*) \leq d(q_{n+1},T(q_n,r_n)) + d(T(q_n,r_n),s^*)$$
$$= d(T(q_n,r_n),T(s^*,t^*)) + \delta_n$$
$$\leq \alpha_1 d(T(s^*,t^*),q_n) + \beta_1 d(T(q_n,r_n),s^*) + \delta_n$$
$$\leq \alpha_1 d(q_n,s^*) + \beta_1 d(T(q_n,r_n),q_n) + \beta_1 d(q_n,s^*) + \delta_n$$
$$= \alpha_1 d(q_n,s^*) + \beta_1 d(q_n,s^*) + \beta_1 d(T(q_n,r_n),q_n) + \delta_n$$
$$= (\alpha_1+\beta_1)d(q_n,s^*) + \beta_1\delta_{n-1} + \delta_n.$$

Hence, passing it to limit and applying Lemma 1 for $h \coloneqq \alpha_1 + \beta_1 \in [0,1)$ and for $\delta'_n \coloneqq \delta_n + \beta_1 \delta_{n-1} \to 0$, we have that $\lim_{n\to\infty} q_n = s^*$.

Similarly, using contraction condition (5), we obtain

$$\begin{aligned} d(r_{n+1}, t^*) &\le d\big(r_{n+1}, T(r_n, q_n)\big) + d(T(r_n, q_n), t^*) \\ &= d(T(r_n, q_n), T(t^*, s^*)) + \gamma_n \\ &\le \alpha_2 d(T(t^*, s^*), r_n) + \beta_2 d(T(r_n, q_n), t^*) + \gamma_n \\ &\le \alpha_2 d(r_n, t^*) + \beta_2 d(T(r_n, q_n), r_n) + \beta_2 d(r_n, t^*) + \gamma_n \\ &= \alpha_2 d(r_n, t^*) + \beta_2 d(r_n, t^*) + \beta_2 d(T(r_n, q_n), r_n) + \gamma_n \\ &= (\alpha_2 + \beta_2) d(r_n, t^*) + \gamma_n + \beta_2 \gamma_{n-1} \end{aligned}$$

So, applying limit and using Lemma 1, for $h \coloneqq \alpha_2 + \beta_2 \in [0,1)$ and for $\gamma'_n \coloneqq \gamma_n + \beta_2 \gamma_{n-1} \to 0$, we obtain that $\lim_{n\to\infty} r_n = t^*$, which is the conclusion.

References

Agarwal, R.P., El-Gebeily, M.A. & O'Regan, D. (2008). Generalized contractions in partially ordered metric spaces, *Appl. Anal.*, 87, 1–8.

Berinde, V. Pacurar, M. (2015). A constructive approach to coupled fixed point theorems in metric spaces, *Carpathian J. Math.*31, 277 – 287.

Borcut, M. & Berinde, V. (2012). Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, *Applied Mathematics & Computation*, 218(14), 5929-5936.

Bhaskar, T.G. & Lakshmikantham, V. (2006). Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis*, 65(7), 1379-1393.

Guo, D. & Lakshmikantham, V. (1987). Coupled fixed points of nonlinear operators with applications, *Nonlinear Analysis Theory Methods & Applications*, 11(5), 623-632.

Harder, A.M. & Hicks, T.L. (1988). Stability results for fixed point iteration procedures, *Math. Japon.*33, 693–706.

Imoru, C.O. & Olatinwo, M.O. On the stability of Picard and Mann iteration processes, *Carpathian J. Math.*19(2), 155–160.

Imoru, C.O., Olatinwo, M.O., & Owojori, O.O. (2006). On the stability results for Picard and Mann iteration procedures, *J. Appl. Funct. Differ. Equ.* 1(1), 71–80.

Jachymski, J.R. (1997). An extension of A. Ostrowski's theorem on the round-off stability of iterations, *Aequationes Math.*,53(3), 242–253.

Lakshmikantham V. & Ćirić L. (2009). Coupled fixed point theorems for nonlinear contractions in partially ordered metric space, *Nonlinear Analysis*, 70(12), 4341-4349.

Matthews, S.G. (1994). Partial metric topology. Annals of the New York Academy of Sciences, 728, 183-197.

Nieto, J.J. & López, R.R. (2005). Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, 22, 223–239.

Nieto, J.J. & López, R.R. (2007). Existence and uniqueness and fixed point in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sinica, Engl. Ser.*, 23(12), 2205–2212.

Olatinwo, M.O., Owojori, O.O., & Imoru, C.O. (2006). Some stability results for fixed point iteration processes. *Aust. J. Math. Anal. Appl.*,3(2), 1–7.

Olatinwo, M.O. (2009). Some stability results in complete metric space, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.*, 48, 83–92.

Olatinwo, M.O. (2012). Some stability results for Picard iterative process in uniform space*Vladikavkaz. Mat. Zh.*,12(4), 67–72.

Osilike, M.O. (1995). Stability results for fixed point iteration procedure. *J. Nigerian Math. Soc.*,14, 17–29.

Osilike, M.O. (1996). A stable iteration procedure for quasi-contractive maps, *Indian J. Pure Appl. Math.*,27(1), 25–34.

Osilike, M.O. (1999). A. Udomene, Short proofs of stability results for fixed point iteration procedures for a class of contractive type mappings, *Indian J. Pure Appl. Math.*,30(12), 1229–1234.

Ostrowski, A.M. (1967). The round-off stability of iterations, *Z. Angew.Math. Mech.*,47, 77–81.

Rhoades, B.E. (1990). Fixed point theorems and stability results for fixed point iteration procedures, *Indian J. Pure Appl. Math.*,21, 1–9.

Rhoades, B.E. (1993). Fixed point theorems and stability results for fixed point iteration procedures II, *Indian J. Pure Appl. Math.*,24, 691–703.

Samet, B. & Vetro, C. (2010). Coupled fixed point F -invariant set and fixed point of N-order, *Ann. Func. Anal.*, 1, 46–56.

Timis, I. (2014). Stability of tripled fixed-point iteration procedures for mixed monotone mappings, *Carpathian Math. Publ.*, 6, 377–388.

Van Luong, N. & Xuan Thuan, N. (2011). Coupled fixed points in partially ordered metric spaces and application, *Nonlinear Anal.*, 74, 983–992.